Centrifugal waves

By O. M. PHILLIPS
Mechanical Engineering Department, The Johns Hopkins University, Baltimore, Maryland

(Received 10 June, 1959)

When a hollow circular cylinder with its axis horizontal is partially filled with water and rotated rapidly about its axis, an almost rigid-body motion results with an interior free surface. The motion is analysed assuming small perturbations to a rigid rotation, and a criterion is found for the stability of the motion. This is confirmed experimentally under varying conditions of water depth and angular velocity of the cylinder. The modes of oscillation (centrifugal waves) of the free surface are examined and a frequency equation deduced. Two particular modes are considered in detail, and satisfactory agreement is found with the frequencies observed.

1. Introduction

The term 'centrifugal waves' is a convenient one to describe the wave motion on the free surface of a rapidly rotating liquid, in which the equilibrium pressure distribution is determined primarily by the radial accelerations associated with the rotation. A simple apparatus in which the properties of these waves can be studied is shown in figure 1. A hollow circular transparent plastic cylinder C was supported by roller bearings B and mounted with its axis horizontal. The ends of the cylinder were closed except for a small hole H on the axis at one end through which water could be introduced slowly. The other end was coupled to a variable-speed motor, M.

When the cylinder is partially filled with water and rotated sufficiently rapidly, the water moves with almost rigid-body rotation about a central air core. The existence of the gravity field perpendicular to the axis and the presence of the free surface together impose a steady (time independent) perturbation on the truly rigid body motion, while for certain combinations of angular velocity and depth of water for a cylinder of given length, a variety of surface wave patterns are observed at the interface. These are most pronounced when the depth of water and speed of rotation are such that the half wavelength in the axial direction is an integral submultiple of the cylinder length. One particularly striking series of wave patterns is such that the free surface inside the rapidly rotating cylinder is stationary with respect to the observer. In another series, the wave crests are parallel to the cylinder axis and move circumferentially relative to the cylinder wall.

The wave frequencies and angular velocity of the cylinder were measured with a stroboscopic light. In a typical experiment, the empty cylinder was rotated at a constant angular velocity and water was introduced very slowly by means of a thin tube passing through the hole H. As the depth of water increased slowly, the various modes of oscillation would build up and die away in turn, until, when the depth approached some critical value, the flow would become unstable and collapse suddenly.

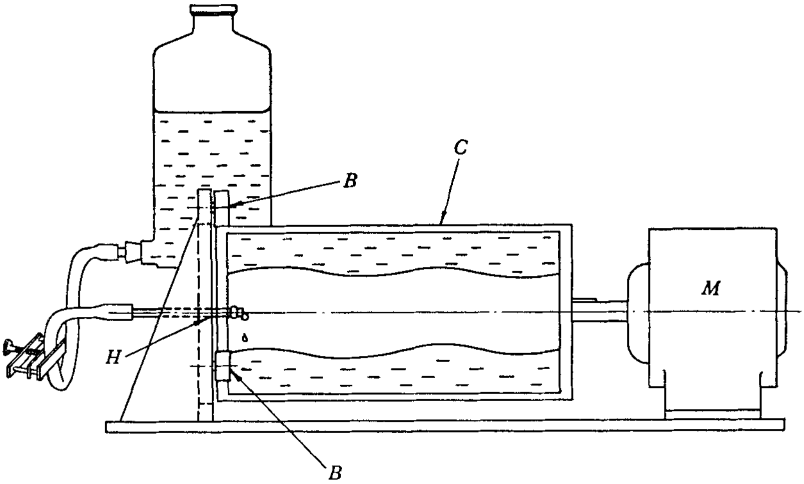

FIGURE 1. The apparatus for observing centrifugal waves. The hollow cylinder C is mounted on roller bearings B and driven by the motor M. Water is introduced through the small axial hole H.

In the following sections, a theory is developed to describe the various motions that can occur. At various points in the analysis, comparisons can be made with observations obtained in this way. Some of these phenomena were first observed by Mr L. Schaaf and Mr R. Drake, and detailed measurements were made with the able assistance of Mr E. C. Crist and Mr D. Meredith. I am also grateful to Mr W. G. Rose for the discussions we have had about this work.

2. The governing equations

Consider the motion inside a rotating circular cylinder of radius a with its axis horizontal, partially filled with a liquid of small viscosity, and suppose that the angular velocity Ω of the cylinder is sufficiently large that the motion of the fluid is almost a rigid-body rotation with a central air core.

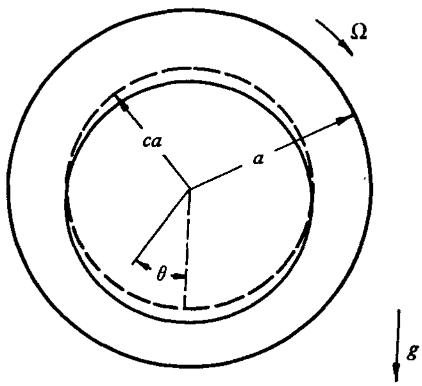

FIGURE 2. Geometry of the free surface.

The momentum and continuity equations for an inviscid fluid* are, expressed in cylindrical polar co-ordinates,

$$\left.\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}+g\cos\theta,\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial z}+\frac{uv}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}-g\sin\theta,\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z},\\
\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z} &= 0,
\end{aligned}\right\} \tag{2.1}$$

where u, v, w represent the velocity components in the directions r, θ, z, respectively, and the direction $\theta = 0$ is taken vertically downwards. If gravity were absent, the steady motion would be that of a rigid body, with

$$\left.\begin{aligned}
&u = 0, \quad v = \Omega r, \quad w = 0,\\
&p = \tfrac{1}{2}\rho\Omega^2(r^2-c^2a^2),
\end{aligned}\right\} \tag{2.2}$$

where a is the radius of the cylinder and ca $(0 < c < 1)$ the radius of the cylindrical free surface. The presence of gravity, acting in a direction perpendicular to the axis, results in a steady perturbation on this rigid-body motion, and, as pointed out already, there may be in addition oscillatory wave disturbances. Therefore, writing

$$\eta = r/a,$$

let

$$\left.\begin{aligned}
u &= \Omega a(u_1+u_2),\\
v &= \Omega a(\eta+v_1+v_2),\\
w &= \Omega a w_2,
\end{aligned}\right\} \tag{2.3}$$

where u_1 and v_1 represent the steady (time-independent) disturbances, and u_2, v_2, w_2 the oscillating (wave) disturbances whose time average is zero. In the presence of gravity, the pressure distribution given by (2.2) is modified by a hydrostatic contribution and by the accompanying displacement of the free surface over which the pressure is constant. We therefore define quantities p_1, p_2 by

$$p = \tfrac{1}{2}\rho\Omega^2a^2\{(\eta^2-c^2)+p_1+p_2\}+\rho g a\eta\cos\theta, \tag{2.4}$$

where p_1 is time independent and the (time) mean value of p_2 is zero.

If the disturbances are sufficiently small (the nature of this requirement is discussed in more detail later), the dimensionless velocities $u_1, u_2, \ldots, w_2$ are small compared with η $(c \leqslant \eta < 1)$ and when (2.3) and (2.4) are substituted into the

* The neglect of the viscous terms in (2.1) is discussed in § 3 below.

equations of motion (2.1), squares and products of the perturbations can be neglected. Thus

$$\left.\begin{aligned}\frac{\partial u_2}{\partial \tau}+\frac{\partial}{\partial \theta}(u_1+u_2)-2(v_1+v_2) &= -\frac{1}{2}\frac{\partial}{\partial \eta}(p_1+p_2),\\ \frac{\partial v_2}{\partial \tau}+2(u_1+u_2)+\frac{\partial}{\partial \theta}(v_1+v_2) &= -\frac{1}{2\eta}\frac{\partial}{\partial \theta}(p_1+p_2),\\ \frac{\partial w_2}{\partial \tau}+\frac{\partial w_2}{\partial \theta} &= -\frac{1}{2}\frac{\partial}{\partial \zeta}(p_1+p_2),\\ \frac{1}{\eta}\frac{\partial}{\partial \eta}\{\eta(u_1+u_2)\}+\frac{1}{\eta}\frac{\partial}{\partial \theta}(v_1+v_2)+\frac{\partial w_2}{\partial \zeta} &= 0,\end{aligned}\right\} \tag{2.5}$$

where $\tau = \Omega t$ and $\zeta = z/a$.

The equations that describe the steady disturbance to the rigid-body motion are obtained by taking the time average of (2.5):

$$\left.\begin{aligned}\frac{\partial u_1}{\partial \theta}-2v_1 &= -\frac{1}{2}\frac{\partial p_1}{\partial \eta},\\ \frac{\partial v_1}{\partial \theta}+2u_1 &= -\frac{1}{2\eta}\frac{\partial p_1}{\partial \theta},\\ \frac{\partial p_1}{\partial \zeta} &= 0,\\ \frac{\partial}{\partial \eta}(\eta u_1)+\frac{\partial v_1}{\partial \theta} &= 0.\end{aligned}\right\} \tag{2.6}$$

The equations for the wave motion are now found by subtracting the set (2.6) from the respective members of (2.5), giving

$$\left.\begin{aligned}\frac{\partial u_2}{\partial \tau}+\frac{\partial u_2}{\partial \theta}-2v_2 &= -\frac{1}{2}\frac{\partial p_2}{\partial \eta},\\ \frac{\partial v_2}{\partial \tau}+\frac{\partial v_2}{\partial \theta}+2u_2 &= -\frac{1}{2\eta}\frac{\partial p_2}{\partial \theta},\\ \frac{\partial w_2}{\partial \tau}+\frac{\partial w_2}{\partial \theta} &= -\frac{1}{2}\frac{\partial p_2}{\partial \zeta},\\ \frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta u_2)+\frac{1}{\eta}\frac{\partial v_2}{\partial \theta}+\frac{\partial w_2}{\partial \zeta} &= 0.\end{aligned}\right\} \tag{2.7}$$

We now turn to consideration of the boundary conditions to be imposed on the sets of equations (2.6) and (2.7). Clearly, at the cylinder wall, both the steady and oscillatory parts of the normal velocity components must vanish, so that

$$u_1 = u_2 = 0 \quad \text{when} \quad \eta = 1. \tag{2.8}$$

Let the free surface, at which the pressure is constant, be given by

$$\eta = c+\delta_1(\theta)+\delta_2(\theta,\zeta,\tau), \tag{2.9}$$

where δ_1 and δ_2 represent dimensionless steady and oscillatory displacements, which are assumed to be small compared with the mean free-surface radius c. Substituting into (2.4) and taking the free-surface pressure as zero, we have

$$2c\frac{g}{\Omega^2 a}\cos\theta+2c(\delta_1+\delta_2)+p_1+p_2 = 0, \tag{2.10}$$

on $\eta = c$, correct to the first order in the infinitesimal wave displacements and to the first order in $g/\Omega^2 a$, the magnitude of the steady disturbance. Taking the mean and oscillating parts of (2.10), we obtain

$$p_1 + 2c\delta_1 = -2c\frac{g}{\Omega^2 a}\cos\theta \tag{2.11}$$

for the steady disturbance, and

$$p_2 + 2c\delta_2 = 0 \tag{2.12}$$

for the wave motion, both when $\eta = c$. The neglect of squares and products of order $(g/\Omega^2 a)^2$ in these boundary conditions is consistent with the neglect of the non-linear disturbance terms in the dynamical equations.

Finally, we have the kinematic boundary condition

$$u = \frac{D\delta}{Dt}$$

at the free surface, and since

$$\frac{D}{Dt} \simeq \frac{\partial}{\partial t} + \frac{\Omega r}{r}\frac{\partial}{\partial \theta},$$

we have

$$u_1 = \frac{\partial \delta_1}{\partial \theta} \tag{2.13}$$

and

$$u_2 = \frac{\partial \delta_2}{\partial \tau} + \frac{\partial \delta_2}{\partial \theta} \tag{2.14}$$

at $\eta = c$, for the steady and wave motions, respectively.

3. The steady disturbance

It is convenient to collect together from the previous section the equations describing the steady disturbance to the rigid body motion, set up by the gravity field. They are

$$\left.\begin{aligned} \frac{\partial u_1}{\partial \theta} - 2v_1 &= -\frac{1}{2}\frac{\partial p_1}{\partial \eta}, \\ \frac{\partial v_1}{\partial \theta} + 2u_1 &= -\frac{1}{2\eta}\frac{\partial p_1}{\partial \theta}, \\ \frac{\partial}{\partial \eta}(\eta u_1) + \frac{\partial v_1}{\partial \theta} &= 0, \end{aligned}\right\} \tag{3.1}$$

with the boundary conditions

$$u_1 = 0 \quad \text{at} \quad \eta = 1, \tag{3.2}$$

$$\left.\begin{aligned} p_1 + 2c\delta_1 &= -2c\frac{g}{\Omega^2 a}\cos\theta \\ u_1 &= \frac{\partial \delta_1}{\partial \theta} \end{aligned}\right\} \quad \text{at} \quad \eta = c. \tag{3.3}$$

The form of these equations suggests the substitutions

$$\begin{aligned} \delta_1 &= \Delta\cos\theta, \\ p_1 &= P(\eta)\cos\theta, \\ u_1 &= \chi(\eta)\sin\theta, \\ v_1 &= \phi(\eta)\cos\theta, \end{aligned}$$

so that

$$\left.\begin{aligned} \chi-2\phi &= -\tfrac{1}{2}P', \\ -\phi+2\chi &= \frac{1}{2\eta}P, \\ \chi+\eta\chi'-\phi &= 0, \end{aligned}\right\} \tag{3.4}$$

and

$$\left.\begin{aligned} \chi = 0 \quad \text{at} \quad \eta = 1, \\ P+2c\Delta = -2c\frac{g}{\Omega^2 a} \quad \text{at} \quad \eta = c, \\ \chi+\Delta = 0 \quad \text{at} \quad \eta = c, \end{aligned}\right\} \tag{3.5}$$

where the primes denote differentiation with respect to η.

Substituting for ϕ from the third equation of (3.4) into the other two, we have

$$-\chi-2\eta\chi' = -\tfrac{1}{2}P',$$

$$\eta\chi+\eta^2\chi' = \tfrac{1}{2}P,$$

and on eliminating P,

$$\eta^2\chi''+3\eta\chi' = 0.$$

Thus

$$\chi = \frac{A}{\eta^2}+B,$$

where A and B are constants to be determined. The first equation of (3.5) gives $B = -A$, so that

$$\chi(\eta) = A\left(\frac{1}{\eta^2}-1\right), \tag{3.6}$$

and from (3.4) it follows that

$$\phi(\eta) = -A\left(\frac{1}{\eta^2}+1\right), \tag{3.7}$$

$$P(\eta) = 2A\eta\left(\frac{3}{\eta^2}-1\right). \tag{3.8}$$

The boundary conditions at $\eta = c$ enable us to determine the constant A and the surface displacement Δ:

$$A = -\frac{c^2}{2}\frac{g}{\Omega^2 a}, \tag{3.9}$$

$$\Delta = \tfrac{1}{2}(1-c^2)\frac{g}{\Omega^2 a}. \tag{3.10}$$

The steady gravity-induced disturbance is thus given by

$$\left.\begin{aligned} u_1 &= -\frac{c^2}{2}\frac{g}{\Omega^2 a}\left(\frac{1}{\eta^2}-1\right)\sin\theta, \\ v_1 &= \frac{c^2}{2}\frac{g}{\Omega^2 a}\left(\frac{1}{\eta^2}+1\right)\cos\theta, \\ p_1 &= -c^2\frac{g}{\Omega^2 a}\eta\left(\frac{3}{\eta^2}-1\right)\cos\theta, \\ \delta_1 &= \frac{1}{2}\frac{g}{\Omega^2 a}(1-c^2)\cos\theta. \end{aligned}\right\} \tag{3.11}$$

The vorticity of this perturbation is of course zero, since the basic flow has constant vorticity. A property of these solutions that is at first sight curious is that the central air core has its axis *below* the axis of the rotating cylinder. However, it can be seen readily that as the fluid moves from points $\theta = 0$ to $\theta = \pi$, it acquires potential energy at the expense of its kinetic energy. The circumferential velocity when $\theta = \pi$ is therefore less than when $\theta = 0$, so that by continuity the thickness of the layer must be greater.

The solutions (3.11) show that at the cylinder surface $\eta = 1$, the circumferential velocity perturbation $v_1 = c^2 g \cos\theta/\Omega^2 a$. If the liquid has viscosity ν, a boundary layer is formed of thickness $(2\nu/\Omega)^{\frac{1}{2}}$, since Ω is the frequency of the perturbation velocity at a fixed point on the rotating cylinder. In order that the above solutions be valid throughout most of the region occupied by the fluid, it is necessary that

$$(2\nu/\Omega)^{\frac{1}{2}} \ll a(1-c),$$

i.e.

$$\frac{\Omega a^2(1-c)^2}{\nu} \gg 2, \tag{3.12}$$

or that the Reynolds number of the layer be large. This condition is invalidated when $1-c$ is very small, and the fluid lies in a thin film inside the cylindrical wall.

In the plane $\theta = \pi$, the gravitational and centrifugal pressure gradients are of opposite sign, and a necessary condition for the stability of the flow is that the net radial pressure gradient be positive. In the absence of wave disturbances, we have from (2.4) and (3.11)

$$p_{\theta=\pi} = \tfrac{1}{2}\rho\Omega^2 a^2\left\{(\eta^2-c^2)+\frac{g}{\Omega^2 a}c^2\left(\frac{3}{\eta}-\eta\right)\right\}-\rho g a\eta$$

$$= \tfrac{1}{2}\rho\Omega^2 a^2\left\{(\eta^2-c^2)+\frac{g}{\Omega^2 a}\left[\frac{3c^2}{\eta}-(2+c^2)\eta\right]\right\},$$

so that a necessary condition for stability is that

$$\left(\frac{\partial p}{\partial \eta}\right)_{\theta=\pi} = \tfrac{1}{2}\rho\Omega^2 a^2\left\{2\eta-\frac{g}{\Omega^2 a}\left[\frac{3c^2}{\eta^2}+(2+c^2)\right]\right\} > 0, \tag{3.13}$$

for $c+\delta_1 \leqslant \eta \leqslant 1$. This condition is strictest when

$$\eta = c+\delta_1 = c-\frac{1}{2}\frac{g}{\Omega^2 a}(1-c^2),$$

from (3.11), so that we require

$$2c-\frac{g}{\Omega^2 a}(1-c^2)-\frac{g}{\Omega^2 a}(5+c^2) > 0,$$

approximately, or

$$\frac{\Omega^2 a}{g} > \frac{3}{c}. \tag{3.14}$$

This condition states that if the flow is to be stable, the value of $\Omega^2 a/g$ cannot be less than $3/c$ (except when $1-c$ is very small, and the flow is viscosity controlled). However, instability and collapse of the flow may occur at values of

$\Omega^2 a/g$ greater than $3/c$ if wave motions of large amplitude are present in addition. This lower limit is illustrated in figure 3, together with some experimental points giving measured values of c for collapse at various values of the parameter $\Omega^2 a/g$. It will be seen that the observational points all lie above the theoretical curve. As mentioned in the introduction, these points were obtained by holding the parameter $\Omega^2 a/g$ constant, and decreasing c from unity by the slow addition of water through the hole H. Values of c at collapse were closest to the theoretical value when $\Omega^2 a/g$ was such that, for the cylinder of given length, there

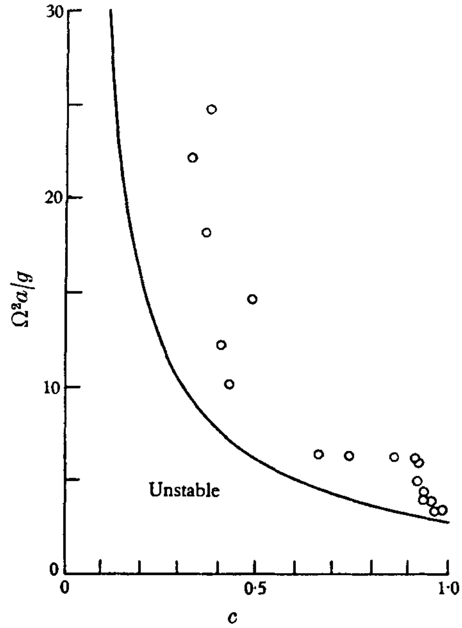

FIGURE 3. The stability condition.

were no large-amplitude wave modes present as c approached its critical value. If there were such waves present, as was usually the case, the collapse occurred prematurely at a larger value of c, because of a momentary unbalance induced by the waves. Near $c = 1$, it is likely that non-linear effects may be significant, since for marginal stability the parameter $\Omega^2 a/g$ is approximately 3, which is not an order of magnitude greater than unity.

It is perhaps worth noting that for a certain region above the curve shown in figure 3, the flow is metastable in the sense that for given values of $\Omega^2 a/g$ and c two types of flow are possible. The first is the almost rigid motion discussed here, and the second a 'collapsed' motion in which the water lies at the bottom of the cylinder and moves in a closed eddy with an almost plane, though tilted, free surface. To restore the almost rigid body motion after collapse, a considerably higher angular velocity of the cylinder is required.

4. The wave motion

The equations to be satisfied by the wave motion are conveniently rewritten here from § 2. The momentum and continuity equations are

$$\left.\begin{aligned} \frac{\partial u_2}{\partial \tau}+\frac{\partial u_2}{\partial \theta}-2v_2 &= -\frac{1}{2}\frac{\partial p_2}{\partial \eta}, \\ \frac{\partial v_2}{\partial \tau}+\frac{\partial v_2}{\partial \theta}+2u_2 &= -\frac{1}{2\eta}\frac{\partial p_2}{\partial \theta}, \\ \frac{\partial w_2}{\partial \tau}+\frac{\partial w_2}{\partial \theta} &= -\frac{1}{2}\frac{\partial p_2}{\partial \zeta}, \\ \frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta u_2)+\frac{1}{\eta}\frac{\partial v_2}{\partial \theta}+\frac{\partial w_2}{\partial \zeta} &= 0. \end{aligned}\right\} \tag{4.1}$$

The appropriate boundary conditions are

$$\left.\begin{aligned} &u_2 = 0 \quad \text{when} \quad \eta = 1, \\ &\left.\begin{aligned} p_2+2c\delta_2 &= 0 \\ u_2 &= \frac{\partial \delta_2}{\partial \tau}+\frac{\partial \delta_2}{\partial \theta}, \end{aligned}\right\} \quad \text{when} \quad \eta = c. \end{aligned}\right\} \tag{4.2}$$

Consider disturbances periodic in time and in the θ- and ζ-directions:

$$\left.\begin{aligned} u_2 &= \chi(\eta)\exp i(n\tau+k\zeta+l\theta), \\ v_2 &= \phi(\eta)\exp i(n\tau+k\zeta+l\theta), \\ w_2 &= \psi(\eta)\exp i(n\tau+k\zeta+l\theta), \\ p_2 &= P(\eta)\exp i(n\tau+k\zeta+l\theta), \\ \delta_2 &= \Delta\exp i(n\tau+k\zeta+l\theta), \end{aligned}\right\} \tag{4.3}$$

where l is an integer and no confusion need arise over the use of the same functional symbols, χ, ϕ, etc., that represented corresponding quantities in the analysis for the steady disturbance on the rigid-body motion.

With these substitutions, the governing equations become

$$\left.\begin{aligned} i(n+l)\chi-2\phi &= -\tfrac{1}{2}P', \\ i(n+l)\phi+2\chi &= -\frac{il}{2\eta}P, \\ i(n+l)\psi &= -\tfrac{1}{2}ikP, \\ \frac{\chi}{\eta}+\chi'+\frac{il}{\eta}\phi+ik\psi &= 0, \end{aligned}\right\} \tag{4.4}$$

with

$$\left.\begin{aligned} \chi = 0 \quad &\text{at} \quad \eta = 1, \\ P+2c\Delta = 0 \quad &\text{at} \quad \eta = c, \\ \chi = i(n+l)\Delta \quad &\text{at} \quad \eta = c. \end{aligned}\right\} \tag{4.5}$$

The equations (4.4) can be solved by substituting for ϕ, χ and ψ in the continuity equation, giving a single equation for P, namely

$$P'' + \frac{1}{\eta}P' + \left(\gamma^2 - \frac{l^2}{\eta^2}\right)P = 0, \tag{4.6}$$

where

$$\gamma^2 = \frac{k^2[4-(n+l)^2]}{(n+l)^2}. \tag{4.7}$$

Equation (4.6) is Bessel's equation and since l is integral, the solution is expressible (if $\gamma^2 > 0$) in terms of Bessel functions of the first and second kind,

$$P(\eta) = \alpha J_l(\gamma\eta) + \beta Y_l(\gamma\eta), \tag{4.8}$$

where α and β are constants whose ratio is to be determined.

The remainder of the solution is obtained readily. From the third of equations (4.4),

$$\psi(\eta) = -\frac{k}{2(n+l)}\{\alpha J_l(\gamma\eta) + \beta Y_l(\gamma\eta)\}, \tag{4.9}$$

while from the first two of this set,

$$\begin{aligned}\phi(\eta) = \frac{\gamma}{4[4-(n+l)^2]}\{&\alpha[(2+n+l)J_{l-1}(\gamma\eta) - (2-n-l)J_{l+1}(\gamma\eta)] \\ &+\beta[(2+n+l)Y_{l-1}(\gamma\eta) - (2-n-l)Y_{l+1}(\gamma\eta)]\},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\chi(\eta) = \frac{-i\gamma}{4[4-(n+l)^2]}\{&\alpha[(2+n+l)J_{l-1}(\gamma\eta) + (2-n-l)J_{l+1}(\gamma\eta)] \\ &+\beta[(2+n+l)Y_{l-1}(\gamma\eta) + (2-n-l)Y_{l+1}(\gamma\eta)]\},\end{aligned} \tag{4.11}$$

where we have used the relations

$$\left.\begin{aligned}\frac{1}{\gamma}\frac{d}{d\eta}J_l(\gamma\eta) &= \frac{l}{\gamma\eta}J_l(\gamma\eta) - J_{l+1}(\gamma\eta), \\ \frac{2lJ_l(\gamma\eta)}{\gamma\eta} &= J_{l-1}(\gamma\eta) + J_{l+1}(\gamma\eta),\end{aligned}\right\} \tag{4.12}$$

and similar expressions for the Y-functions.

The ratio α/β is determined by the first of the boundary conditions (4.5), and from (4.11) we have immediately

$$\frac{\alpha}{\beta} = -\frac{(2+n+l)Y_{l-1}(\gamma) + (2-n-l)Y_{l+1}(\gamma)}{(2+n+l)J_{l-1}(\gamma) + (2-n-l)J_{l+1}(\gamma)}. \tag{4.13}$$

The second and third members of the set (4.5) determine the frequency equation $n = n(k, l, c)$. Eliminating Δ,

$$\chi + \frac{i(n+l)}{2c}P = 0 \quad \text{at} \quad \eta = c, \tag{4.14}$$

and substitution from the expressions above gives, after some algebra,

$$\begin{aligned}&\frac{\left(\dfrac{l+n}{l} - \dfrac{1}{2-n-l}\right)J_{l-1}(\gamma c) + \left(\dfrac{l+n}{l} - \dfrac{1}{2+n+l}\right)J_{l+1}(\gamma c)}{(2+n+l)J_{l-1}(\gamma) + (2-n-l)J_{l+1}(\gamma)} \\ &\qquad = \frac{\left(\dfrac{l+n}{l} - \dfrac{1}{2-n-l}\right)Y_{l-1}(\gamma c) + \left(\dfrac{l+n}{l} - \dfrac{1}{2+n+l}\right)Y_{l+1}(\gamma c)}{(2+n+l)Y_{l-1}(\gamma) + (2-n-l)Y_{l+1}(\gamma)},\end{aligned} \tag{4.15}$$

an implicit relation between n, k, l and c.

Of the many possible wave motions, one of the most striking is the one specified by $l = 1$, $n = 0$. The free surface is stationary with respect to the observer and stands like a sinusoidally flexed transparent cylinder within the rapidly rotating outer case. With $l = 1$, $n = 0$ (4.15) reduces to

$$\{3Y_0(\gamma)+Y_2(\gamma)\}J_2(\gamma c)-\{3J_0(\gamma)+J_2(\gamma)\}Y_2(\gamma c) = 0, \qquad (4.16)$$

where now $\gamma = k\sqrt{3}$. The allowable axial wave-numbers k for this wave mode are given by the roots of this equation and are illustrated as functions of c in figure 4. It is evident that k is multiple valued; an infinite number of wave-numbers k with

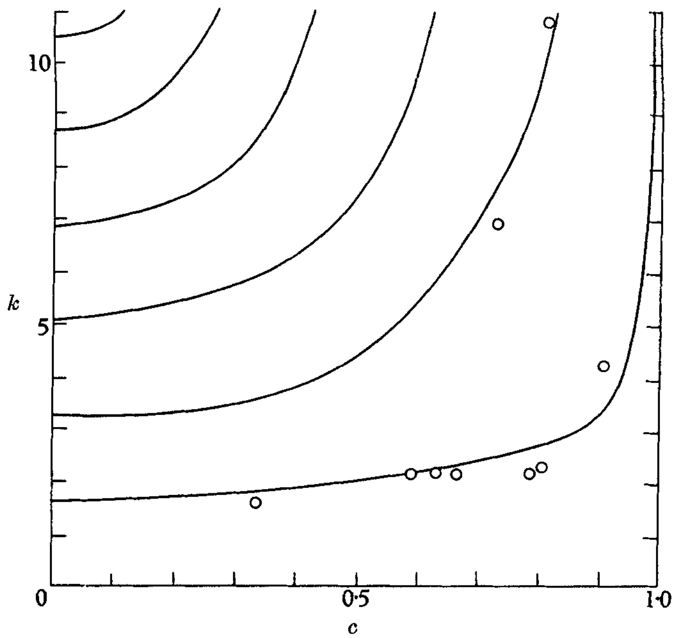

FIGURE 4. Allowable wave-numbers for stationary waves ($n = 0$) as a function of c.

$n = 0$ being possible for a given value of c. These waves were excited in the apparatus only when the wavelength was an integral submultiple of twice the cylinder length, so that only a few distinct modes of oscillation were observed. The agreement between the wave-numbers of the modes that were excited and the theoretical predictions are quite good, points being found near the two lowest branches of figure 4. For small values of c we have seen that the flow becomes unstable even for large values of $\Omega^2 a/g$, and the minimum attainable value of c for almost rigid body rotation in our apparatus was a little less than 0·4.

If $\gamma^2 < 0$, a similar frequency equation can be derived without difficulty, involving Bessel functions of purely imaginary argument. If $\gamma = 0$, that is, if either $k = 0$ or $(l+n)^2 = 4$, the above solutions become degenerate. The case $k = 0$ is more interesting physically since it represents a two-dimensional motion in which the crests are parallel to the cylinder axis and move in a circumferential direction.

When $\gamma = 0$, equation (4.6) reduces to

$$P'' + \frac{1}{\eta}P' - \frac{l^2}{\eta^2}P = 0,$$

whence
$$P(\eta) = A\eta^l + B\eta^{-l}, \tag{4.17}$$

where A and B are constants. If $k = 0$, we have immediately from (4.4),

$$\psi(\eta) = 0$$

$$\phi(\eta) = \frac{Al\eta^{l-1}}{2(2-n-l)} - \frac{Bl\eta^{-l-1}}{2(2+n+l)}, \tag{4.18}$$

$$\chi(\eta) = \frac{-iAl\eta^{l-1}}{2(2-n-l)} - \frac{iBl\eta^{-l-1}}{2(2+n+l)}. \tag{4.19}$$

The first boundary condition (4.5) requires

$$\frac{A}{B} = -\frac{2-n-l}{2+n+l},$$

so that

$$\left.\begin{aligned} P(\eta) &= \frac{A}{2-n-l}\{(2-n-l)\,\eta^l - (2+n+l)\,\eta^{-l}\}, \\ \phi(\eta) &= \frac{Al}{2(2-n-l)}\{\eta^{l-1} + \eta^{-l-1}\}, \\ \chi(\eta) &= \frac{-iAl}{2(2-n-l)}\{\eta^{l-1} - \eta^{-l-1}\}. \end{aligned}\right\} \tag{4.20}$$

The frequency equation follows from the boundary conditions at $\eta = c$. From (4.5),

$$\chi + \frac{i(n+l)}{2c}P = 0 \quad \text{at} \quad \eta = c,$$

whence, on substitution from (4.16),

$$n^2(1+c^{2l}) + 2n\{(1+l) - (1-l)\,c^{2l}\} + l\{(1+l) - (1-l)\,c^{2l}\} = 0. \tag{4.21}$$

This frequency equation can be obtained alternatively from (4.15) by a limiting process as $\gamma \to 0$, $(l+n)^2 \neq 4$. The roots of (4.21) are illustrated in figure 5 as functions of c for the case $l = 1$, which was the mode excited most readily in our apparatus. For a given value of c there are two possible wave frequencies, one greater and one less than the rotation frequency of the cylinder. The slow waves were observed most commonly, and their frequencies agreed well with the theoretical curve. The occurrence of these waves was often a prelude to premature collapse. On two occasions only were the high-frequency waves observed and both times they occurred in combination with one of the other modes. Presumably the high-frequency waves are more heavily damped by viscosity, and this may account for the relative difficulty in generating them.

It is interesting to notice that when $c = 1-\epsilon$, $\epsilon \ll 1$, equation (4.21) gives approximately

$$n = -l(1 \pm \epsilon^{\frac{1}{2}}),$$

where the relative signs of l and n determine the direction of wave motion. In a frame of reference rotating with the cylinder,

$$n = \mp l\epsilon^{\frac{1}{2}},$$

or in dimensional form, with frequency $\sigma = \Omega n$ and layer depth $d = \epsilon a$,

$$\sigma = \pm l\left(\frac{\Omega^2}{a}\right)^{\frac{1}{2}} d^{\frac{1}{2}}.$$

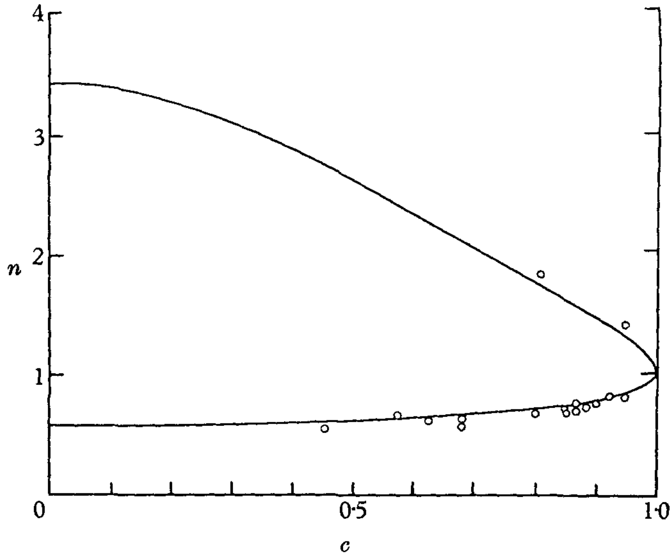

FIGURE 5. The frequency of circumferential waves with one node.

Since the wavelength $\lambda = 2\pi a/l$, and the wave velocity $V = \sigma\lambda/2\pi$, we have

$$V = (\Omega^2 a)^{\frac{1}{2}} d^{\frac{1}{2}}. \tag{4.22}$$

This is seen to be exactly analogous to the velocity of gravity waves under the long wave approximation ($\lambda \gg d$), where

$$V = g^{\frac{1}{2}} d^{\frac{1}{2}}.$$

The gravitational acceleration g is here replaced by the radial acceleration $\Omega^2 a$ in the rotating system.

This work was supported by the Office of Naval Research under contract Nonr 38(248).